SIRPY · Problem 4 — The Gross–Pitaevskii Equation

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The Equation Where Even the Eigenvalue Is Unknown

An ongoing series. We are building a public library of problems SIRPY has solved — each entry is one problem, the exact input we handed the solver, the exact output it returned, and an honest read of what happened. No slides. No hand-waving. Numbers you can check.

Three unknowns at once

The problems in this library have been climbing a ladder. Problem 1 recovered a missing initial slope. Problem 3 recovered an unknown *functional* — a coefficient defined as an integral of the solution itself.

This one asks for something harder still. Here we do not know the solution, we do not know its initial slope, **and we do not know the constant that appears in the equation.** All three must be determined together, subject to two boundary conditions and a normalization constraint.

The equation is the one-dimensional, time-independent **Gross–Pitaevskii equation** — the standard model for a Bose–Einstein condensate in a harmonic trap:

$$-y'' + x^2 y + y^3 \;=\; \mu\, y, \qquad y(-5) = 0, \qquad y(5) = 0, \qquad \int_{-5}^{5} |y|^2 \, dt = 1. \tag{1}$$

Read what is being asked. The trap is the $x^2 y$ term. The atoms repel each other through the cubic term y^3, which makes the problem nonlinear. And μ — the chemical potential — is **not given**. It is an eigenvalue: a number that must be discovered along with the solution, because only for particular values of μ does a solution satisfying all three conditions exist at all.

The normalization $\int_{-5}^{5} |y|^2 = 1$ is what pins it down physically: it says the condensate contains exactly one unit of probability. Without it, any multiple of a solution would also be a solution and μ would be undetermined. With it, the problem has structure — but it is now a **nonlinear eigenvalue problem with a nonlocal side condition**, which is a genuinely awkward object for tools built around "march from here to there."

What we handed the solver

```
from sirpy import solve_bvp, plot_verification

r_gp = solve_bvp(order=2, f_str="y**3 - (mu - x**2)*y", x0=-5, x1=5,
                 bc=["y(-5)=0",
                     "y(5)=0",
                     "integral(abs(y)**2,-5,5)=1"],
                 eigen="mu",
                 iterations=30, verbose=True)

print("mu =", r_gp.mu)
plot_verification(r_gp, save="verify_gp.png", show=True)
```

Look at the `bc` list. The three conditions are written the way they are written on paper: two endpoint values, and an integral constraint stated as an integral. The unknown eigenvalue is named with `eigen="mu"` and referenced directly in the equation string. Nothing is reformulated into a first-order system, no mesh is designed, and no initial guess for μ is supplied.

That is the entire program.

What came back

It changed strategy, and said so

```
INFO: single shooting could not solve this problem, so sirpy escalated
      to MULTIPLE SHOOTING automatically.
INFO: segments refined until mu stopped changing; settled at n_shoot = 16.
```

SIRPY's first approach failed. Rather than returning a degraded answer or a silent failure, it escalated to multiple shooting, refined the number of segments until the eigenvalue stopped moving, settled at 16 — and reported every step of that decision.

It explained where μ came from

```
INFO: mu seeded from the LINEAR limit (the nonlinear terms dropped):
      level 0 of -y'' + (x**2)y = mu*(1)y gives mu = 0.999961, with its
      eigenfunction as the starting profile. A nonlinear problem has
      several branches and this seed chooses one; pass guess=/guess_fn=
      to steer elsewhere.
```

To find an unknown eigenvalue you need somewhere to start. SIRPY dropped the nonlinear term, solved the resulting *linear* problem — the quantum harmonic oscillator, whose ground state is a textbook result — and used its eigenvalue $\mu \approx 1$ and eigenfunction as the starting profile.

Then it told you plainly that this choice selects **one branch among several**, and gave you the knob to look elsewhere. That disclosure matters: for a nonlinear eigenvalue problem, "the answer" is not a well-posed notion, and a solver that pretends otherwise is misleading you.

The result

Table 1: Solver output and verification

Quantity	Value
Chemical potential μ	**1.3834778887096502**
Normalization $\int_{-5}^{5} \lvert y\rvert^2 dt$	**1.0000000000000004**
Residual by substitution	6.27×10^{-15}
Shooting segments	16 (refined until μ stabilized)
Iterations	30

Is it right? Three independent checks

A solution to a nonlinear eigenvalue problem is exactly the kind of result that deserves suspicion. It satisfies conditions that were chosen to make it exist. So we check it three ways, none of which relies on trusting the solver.

1. Substitution into the equation

```
INFO: verification by substitution - VERIFIED: satisfies the equation
      to 6.27e-15 on [-5, 5]. Checked by exact coefficient
      differentiation at interior points, not at the interface nodes.
```

Because SIRPY returns the solution as an analytic object, it can be differentiated exactly and put back into Equation 1. The defect across the whole interval is 6.27×10^{-15} — machine precision. The check is performed at interior points, deliberately away from the interface nodes where agreement would be uninformative.

2. The normalization constraint

The third condition is not a boundary value; it is an integral over the entire domain, and it is the physically meaningful one. Computed independently from the returned solution:

$$\int_{-5}^{5} |y(t)|^2 \, dt \;=\; 1.0000000000000004$$

Unity to fifteen decimal places. The constraint was not approximately imposed — it holds to the last bit the machine can represent.

3. The eigenvalue is stable under refinement

Running the same problem at a lower resolution (20 iterations instead of 30) returns:

Table 2: Eigenvalue under refinement

Iterations	μ
20	1.38347788867945
30	1.3834778887096502

Two independent resolutions agree to roughly ten significant figures. An eigenvalue that drifts as you refine is an artifact; one that sits still is a property of the problem.

And a physical sanity check

Drop the interaction term and the equation becomes the quantum harmonic oscillator, whose ground state has $\mu = 1$ exactly — which is what SIRPY found as its seed (0.999961). Switching the repulsive interaction back on should push the chemical potential **upward**, because repelling atoms spread against the trap and cost energy. SIRPY returns $\mu = 1.3835$.

The direction is right, and the magnitude is physically sensible for this coupling. The mathematics and the physics agree.

The figures

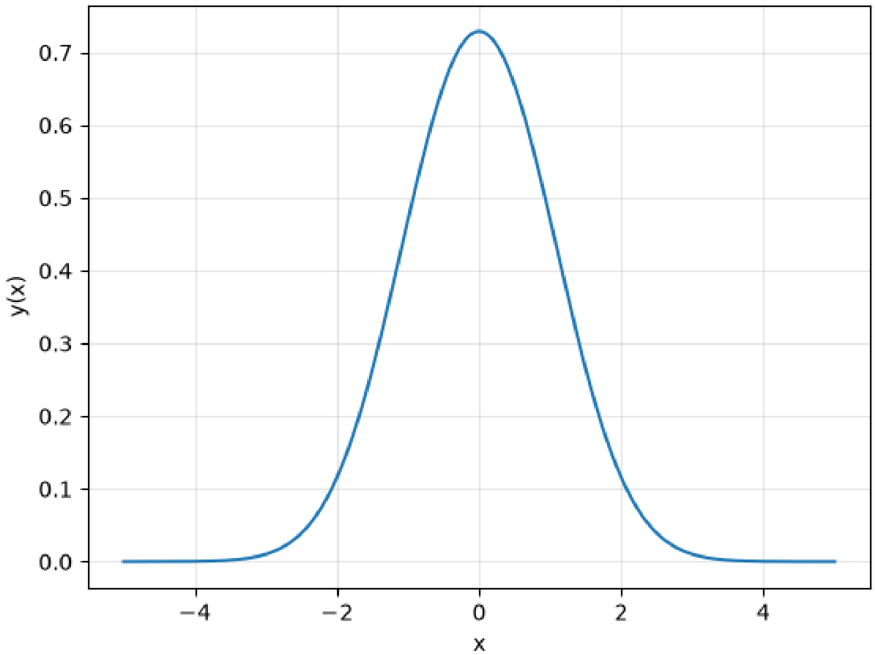

Figure 1: The solution on $[-5, 5]$: the condensate ground-state profile, vanishing at both walls of the trap and normalized to unit probability.

What this problem shows

- **Three unknowns resolved together.** The solution, its initial slope, and the eigenvalue μ — none supplied, all determined, subject to two boundary conditions and an integral constraint.
- **The constraint is stated, not encoded.** `"integral(abs(y)**2,-5,5)=1"` is written the way the condition reads. No reformulation, no penalty function, no manual outer loop.
- **The solver adapts and reports.** Single shooting failed; SIRPY escalated to multiple shooting, refined to 16 segments, and disclosed both the failure and the fix.
- **It shows its assumptions.** Where the eigenvalue seed came from, that the seed selects one branch, and how to choose another.
- **Verified three ways** — substitution to 6×10^{-15}, normalization to fifteen decimals, and eigenvalue stability across refinement.

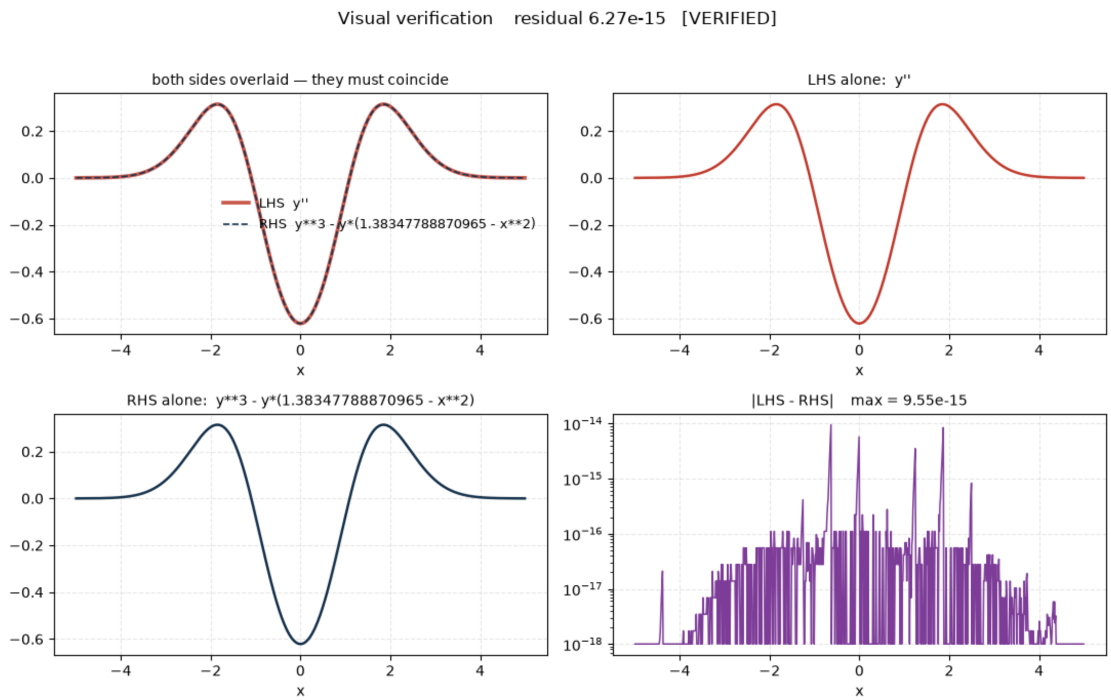

Figure 2: Verification: both sides of the equation, compared across the interval. The solution is substituted back into Equation 1 and the defect measured directly.

A note on scope

This is a nonlinear eigenvalue problem, so it admits multiple branches. The result above is the branch reached from the linear-limit seed, and we do not claim it is the only solution. SIRPY says so itself in its output, and provides `guess=` / `guess_fn=` to reach others. What is claimed here is narrow and checkable: this solution satisfies the equation to 6.27×10^{-15}, meets both boundary conditions, and satisfies the normalization to 1.0000000000000004.

The problem library

This entry is part of the SIRPY Problem Library — an ongoing, public catalog of mathematical systems solved and verified by SIRPY. Each entry contains the problem as posed, the exact input given to the solver, the solver's own reported output, and the verification residual.

The library is maintained in two places:

- **GitHub** — github.com/Mitra-Institute-of-Education-MIE/sirpy-problem-library
- **Web** — mitrainstituteofeducation.org/sirpy

Both are updated as new problems are solved and verified.

Submit a problem

Have an ordinary differential equation you believe is challenging, unusual, or simply beautiful? We would like to see it. We will run it through SIRPY and report honestly what happened — if it solves it, how; if it struggles, why; if it reaches a current limit, we will say so.

Contact: info@mitrainstituteofeducation.org · mitrainstituteofeducation.org/contact

Please only send problems you are free to share publicly.

SIRPY is in final validation prior to its first public release. Every line quoted above is verbatim from a single run of the solver; nothing is reproduced from memory.